Normally, for analysis of experiments using Special Relativity between inertial reference frames, the simple one dimensional formulas to transform position, velocity and acceleration are adequate. However, it will be useful to have more general two dimensional transformations. To develop these formulas, consider Figure 1.

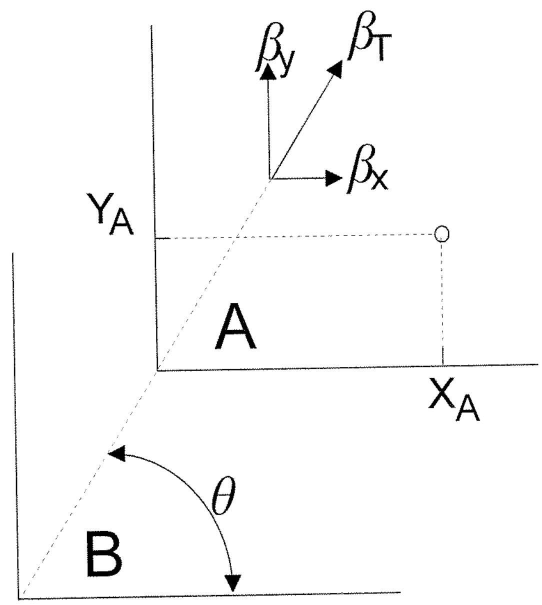

Figure 1

Reference frame B is stationary and reference frame A is passing with total velocity β_T, which can be resolved into components β_X and β_Y. The origins of the two reference frame are exactly on top of each other at times $t_A = t_B = 0$. A random point at position x_A, y_A is shown. Frame A will contract in the direction of velocity β_T and will appear distorted as is shown in Figure 2.

The dotted lines indicate frame A as viewed by frame A. The solid lines show the effects of length contraction on frame A as seen by frame B. The origin of frame A has not changed location. The location of the point x_A, y_A contracts a distance d_T in a direction parallel to the direction of β_T. Distance d_T is made of two components, d_X and d_Y ($d_T = d_X + d_Y$), which are the distances that the coordinates x_A and y_A move (parallel to the direction of β_T).

$$\cos\theta = \left(\frac{\beta_X}{\beta_T}\right) \tag{1a}$$

$$\sin\theta = \left(\frac{\beta_Y}{\beta_T}\right) \tag{1b}$$

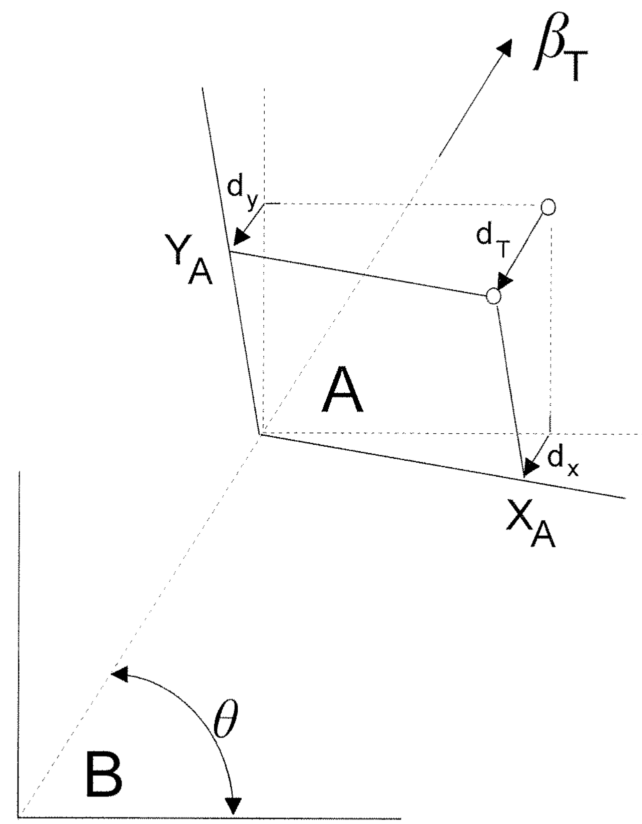

Figure 2

The length of d_X can be determined from Figure 3.

$$d'_x = x_A \sqrt{1-\beta_T^2}\ \cos\theta$$

$$dx = x_A\left(\frac{\beta_X}{\beta_T}\right)(1-\sqrt{1-\beta_T^2}\,) \tag{2}$$

Similarly, the length of d_Y can be determined from Figure 4.

$$d'_Y = y_A \sqrt{1-\beta_T^2}\ \sin\theta$$

$$d_Y = y_A\left(\frac{\beta_Y}{\beta_T}\right)(1-\sqrt{1-\beta_T^2}\,) \tag{3}$$

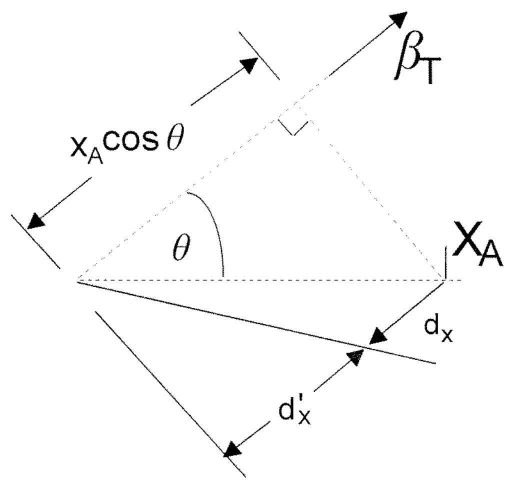

Figure 3

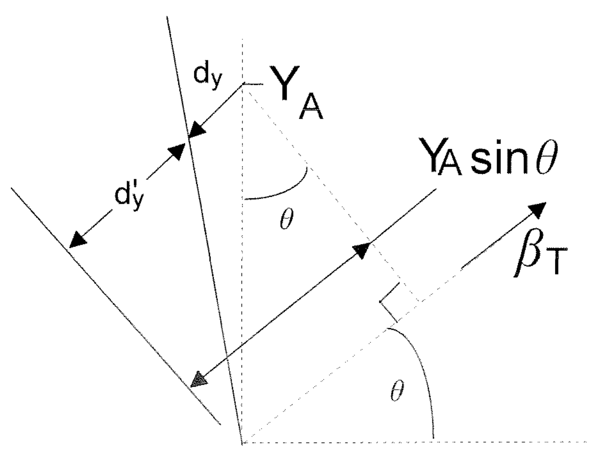

Figure 4

The quantities d_X and d_Y relate to the frame B axes as shown in Figure 5.

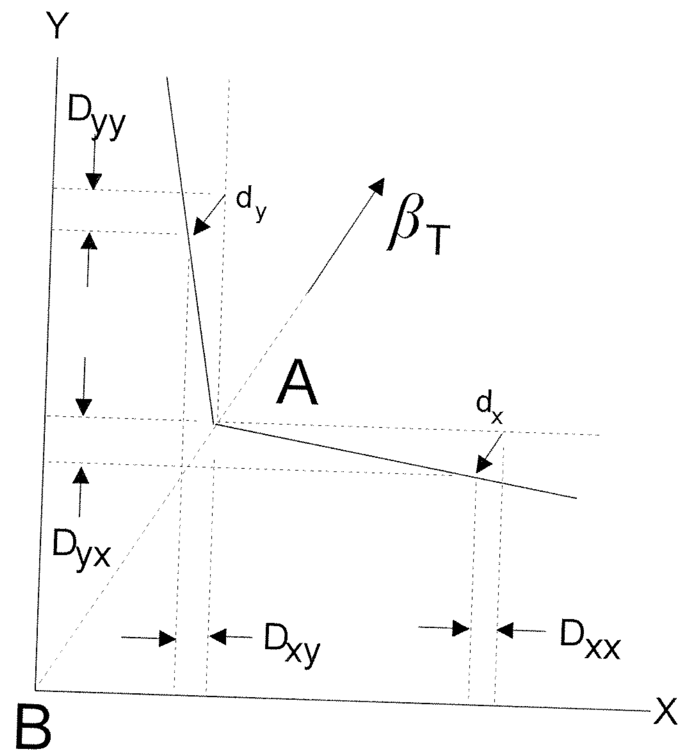

Figure 5

The frame A contraction d_X changes the frame B x-axis position by an amount D_{XX} and simultaneously changes the frame B y-axis position by an amount D_{YX}. The frame A contraction d_Y changes the frame B x-axis position by an amount D_{XY} and simultaneously changes the frame B y-axis position by an amount D_{YY}.

$$D_{XX} = d_X \cos\theta = x_A\left(\frac{\beta_X^2}{\beta_T^2}\right)(1 - \sqrt{1-\beta_T^2}) \qquad (4a)$$

$$D_{XY} = d_Y \cos\theta = y_A\left(\frac{\beta_Y \beta_X}{\beta_T^2}\right)(1 - \sqrt{1-\beta_T^2}) \qquad (4b)$$

$$D_{YX} = d_X \sin\theta = x_A\left(\frac{\beta_Y\beta_X}{\beta_T^2}\right)(1 - \sqrt{1-\beta_T^2}\,) \qquad (4c)$$

$$D_{YY} = d_Y \sin\theta = y_A\left(\frac{\beta_Y^2}{\beta_T^2}\right)(1 - \sqrt{1-\beta_T^2}\,) \qquad (4d)$$

The clock reading in frame B is t_B. The position of the point x_A, y_A in the B reference frame is:

$$x_B - c\beta_X t_B = x_A - D_{XX} - D_{YX} \qquad (5a)$$

$$y_B - c\beta_Y t_B = y_A - D_{XY} - D_{YY} \qquad (5b)$$

The structure of (5) accounts for the length contraction of the frame A coordinates as seen by frame B. The final result is:

$$x_B - c\beta_X t_B = x_A[1 - \left(\frac{\beta_X^2}{\beta_T^2}\right)(1 - \sqrt{1-\beta_T^2}\,)] - y_A\left(\frac{\beta_Y\beta_X}{\beta_T^2}\right)(1 - \sqrt{1-\beta_T^2}\,) \qquad (6a)$$

$$y_B - c\beta_Y t_B = y_A[1 - \left(\frac{\beta_Y^2}{\beta_T^2}\right)(1 - \sqrt{1-\beta_T^2}\,)] - x_A\left(\frac{\beta_Y\beta_X}{\beta_T^2}\right)(1 - \sqrt{1-\beta_T^2}\,) \qquad (6b)$$

Equation (6) gives the frame B coordinates (frame B view) of a point in frame A. Similarly, a position in frame B (frame A view) will have the following frame A coordinates.

$$x_A + c\beta_X t_A = x_B[1 - \left(\frac{\beta_X^2}{\beta_T^2}\right)(1 - \sqrt{1-\beta_T^2}\,)] - y_B\left(\frac{\beta_Y\beta_X}{\beta_T^2}\right)(1 - \sqrt{1-\beta_T^2}\,) \qquad (7a)$$

$$y_A + c\beta_Y t_A = y_B[1 - \left(\frac{\beta_Y^2}{\beta_T^2}\right)(1 - \sqrt{1-\beta_T^2}\,)] - x_B\left(\frac{\beta_Y\beta_X}{\beta_T^2}\right)(1 - \sqrt{1-\beta_T^2}\,) \qquad (7b)$$

Both (6) and (7) require β_X and β_Y to be constant (inertial reference frame). The clock reading of point x_A, y_A as seen by frame B can be found using Figure 6. The length of line d_{cl} is:

$$d_{cl} = \sqrt{x_A^2 + y_A^2}\,\cos(\theta - \gamma) \qquad (8)$$

$$\sin\gamma = \left(\frac{y_A}{\sqrt{x_A^2 + y_A^2}}\right) \qquad \cos\gamma = \left(\frac{x_A}{\sqrt{x_A^2 + y_A^2}}\right)$$

The time that frame B sees on the point x_A, y_A clock is t_{AP}:

$$t_{AP} = t_B\sqrt{1-\beta_T^2} - \left(\frac{d_{cl}\beta_T}{c}\right) \qquad (9)$$

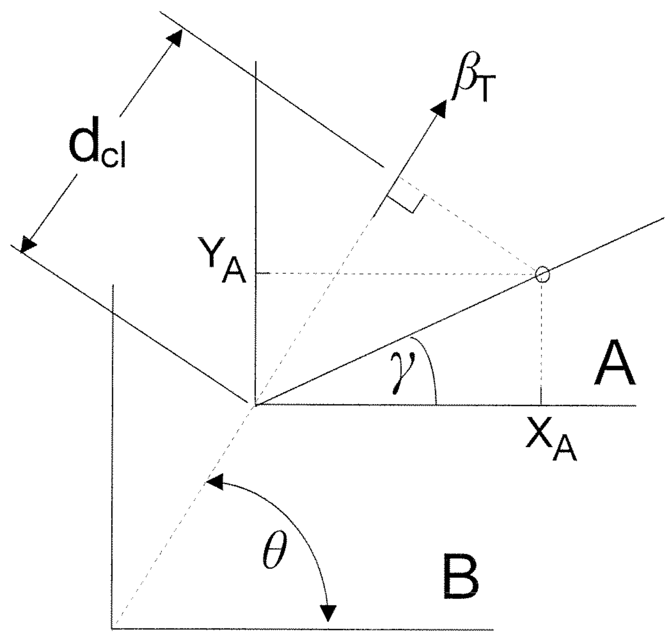

Figure 6

The resulting relationship between clock readings is:

$$t_{AP} = t_B\sqrt{1-\beta_T^2} - \left(\frac{x_A\beta_X}{c}\right) - \left(\frac{y_A\beta_Y}{c}\right) \tag{10}$$

Equation (10) is from the point of view of frame B. An observer in frame B with clock reading t_B who is standing next to frame A point x_A, y_A would see clock reading t_{AP} on that frame A clock. He would then have to use (6) or (7) to know his coordinates x_B and y_B. Equation (10) can be combined with (6) to get:

$$t_{AP} = \left(\frac{t_B - \left(\frac{x_B\beta_X}{c}\right) - \left(\frac{y_B\beta_Y}{c}\right)}{\sqrt{1-\beta_T^2}}\right) \tag{11}$$

If an observer in frame B stands at point x_B, y_B and has clock reading t_B, then the frame A clock at point x_A, y_A on top of his position reads t_{AP}.

If an observer in frame B sees an object at point x_A, y_A move with a velocity β_{XB} in the x-direction and β_{YB} in the y-direction, the object velocities seen by frame A would be β_{XA} in the x-direction and β_{YA} in the y-direction. These velocities would be found by differentiating (7) and (11).

$$\beta_{XA} = [\beta_{XB}(1 - \left(\frac{\beta_X^2}{\beta_T^2}\right)K_S) - \beta_{YB}\left(\frac{\beta_Y\beta_X}{\beta_T^2}\right)K_S]K_{TB} - \beta_X \tag{12}$$

$$\beta_{YA} = [\beta_{YB}(1 - \left(\frac{\beta_Y^2}{\beta_T^2}\right)K_S) - \beta_{XB}\left(\frac{\beta_Y\beta_X}{\beta_T^2}\right)K_S]K_{TB} - \beta_Y \tag{13}$$

$$K_{TB} = \left(\frac{\sqrt{1-\beta_T^2}}{1 - \beta_X\beta_{XB} - \beta_Y\beta_{YB}}\right)$$

$$K_S = 1 - \sqrt{1-\beta_T^2}$$

Equations (12) and (13) can also be rewritten from the point of view of frame A as:

$$\beta_{XB} = [\beta_{XA}(1 - \left(\frac{\beta_X^2}{\beta_T^2}\right)K_S) - \beta_{YA}\left(\frac{\beta_Y\beta_X}{\beta_T^2}\right)K_S]K_{TA} + \beta_X \tag{14}$$

$$\beta_{YB} = [\beta_{YA}(1 - \left(\frac{\beta_Y^2}{\beta_T^2}\right)K_S) - \beta_{XA}\left(\frac{\beta_Y\beta_X}{\beta_T^2}\right)K_S]K_{TA} + \beta_Y \tag{15}$$

$$K_{TA} = \left(\frac{\sqrt{1-\beta_T^2}}{1 + \beta_X\beta_{XA} + \beta_Y\beta_{YA}}\right)$$

Defining relativistic acceleration as A = (Newtonian acceleration)/c, if point x_A, y_A has an acceleration A_{XB} in the x-direction and A_{YB} in the y-direction (as seen by frame B), the accelerations seen by frame A would be A_{XA} in the x-direction and A_{YA} in the y-direction. These accelerations would be found by differentiating (12) and (13).

$$A_{XA} = \{[A_{XB} + \beta_{XB}K_B](1 - \left(\frac{\beta_X^2}{\beta_T^2}\right)K_S) - [A_{YB} + \beta_{YB}K_B]\left(\frac{\beta_X\beta_Y}{\beta_T^2}\right)K_S\}K_{TB}^2 \tag{16}$$

$$A_{YA} = \{[A_{YB} + \beta_{YB}K_B](1 - \left(\frac{\beta_Y^2}{\beta_T^2}\right)K_S) - [A_{XB} + \beta_{XB}K_B]\left(\frac{\beta_X\beta_Y}{\beta_T^2}\right)K_S\}K_{TB}^2 \tag{17}$$

$$K_B = \left(\frac{\beta_X A_{XB} + \beta_Y A_{YB}}{1 - \beta_X\beta_{XB} - \beta_Y\beta_{YB}}\right)$$

This equation can also be rewritten to view the experiment as frame A sees it by differentiating (14) and (15):

$$A_{XB} = \{[A_{XA} - \beta_{XA}K_A](1 - \left(\frac{\beta_X^2}{\beta_T^2}\right)K_S) - [A_{YA} - \beta_{YA}K_A]\left(\frac{\beta_X\beta_Y}{\beta_T^2}\right)K_S\}K_{TA}^2 \qquad (18)$$

$$A_{YB} = \{[A_{YA} - \beta_{YA}K_A](1 - \left(\frac{\beta_Y^2}{\beta_T^2}\right)K_S) - [A_{XA} - \beta_{XA}K_A]\left(\frac{\beta_X\beta_Y}{\beta_T^2}\right)K_S\}K_{TA}^2 \qquad (19)$$

$$K_A = \left(\frac{\beta_X A_{XA} + \beta_Y A_{YA}}{1 + \beta_X\beta_{XA} + \beta_Y\beta_{YA}}\right)$$

Anyone using the equations of this article should keep in mind that there are limitations on the variables involved based upon the vector nature of many of the quantities. For example, $\beta_X = 0.9$ and $\beta_Y = 0.9$ are not possible inputs simultaneously because they would result in $\beta_T = 1.273$. β_T cannot be greater than 1.0, so it's two constituents must be chosen accordingly. Other input variables are also under the same constraint. In addition, inputing β variables as exact values of 1.0000 (for example) will lead to misleading results. If β values of this magnitude are desired, then use a value of 0.99 (for example).

Two dimensional positions, velocities and accelerations involve many variables as they transform from one inertial reference frame to another. As an object is observed moving in one reference frame, predicting its movement as observed from another reference frame may not be intuitive. The equations given in this article will be useful in subsequent articles in this series which involve dynamic calculations of relativistic experiments.