

Forces fall into three main categories: static, dynamic and field generated. The study of dynamic force begins with its relationship to momentum.

The General Case

Consider the general case of a force impressed on a mass m . Relative to a stationary reference frame B, this force results in a change in the vector momentum \vec{P}_B of the mass.

$$\vec{P}_B = \frac{mc \vec{\beta}_B}{\sqrt{1 - |\vec{\beta}_B|^2}}$$

$$\vec{P}_B = P_{XB} \vec{i} + P_{YB} \vec{j} \qquad \vec{\beta}_B = \beta_{XB} \vec{i} + \beta_{YB} \vec{j}$$

\vec{i}, \vec{j} - unit vectors in the x, y directions

$F_{XB} = dP_{XB} / dt_B$ - force on mass m in the x-direction

$F_{YB} = dP_{YB} / dt_B$ - force on mass m in the y-direction

$$F_{XB} = \frac{mc}{(1 - \beta_{XB}^2 - \beta_{YB}^2)^{3/2}} \left[(1 - \beta_{YB}^2) A_{XB} + \beta_{XB} \beta_{YB} A_{YB} \right] \quad (20a)$$

$$F_{YB} = \frac{mc}{(1 - \beta_{XB}^2 - \beta_{YB}^2)^{3/2}} \left[(1 - \beta_{XB}^2) A_{YB} + \beta_{XB} \beta_{YB} A_{XB} \right] \quad (20b)$$

$$A_{XB} = \frac{d\beta_{XB}}{dt_B} \quad - \text{(x-direction acceleration of m)/c}$$

$$A_{YB} = \frac{d\beta_{YB}}{dt_B} \quad - \text{(y-direction acceleration of m)/c}$$

The “Force Enhancement” Principle of Special Relativity

Equation (20) can be applied to experiments where force and momentum are involved. For example, consider the experiment of Figure 7, where a stationary reference frame B is

pushing on a mass m with forces F_{xB} and F_{yB} . Mass m is stationary in a reference frame A that is moving with x-direction velocity β_x (the y-direction velocity is $\beta_y = 0$).

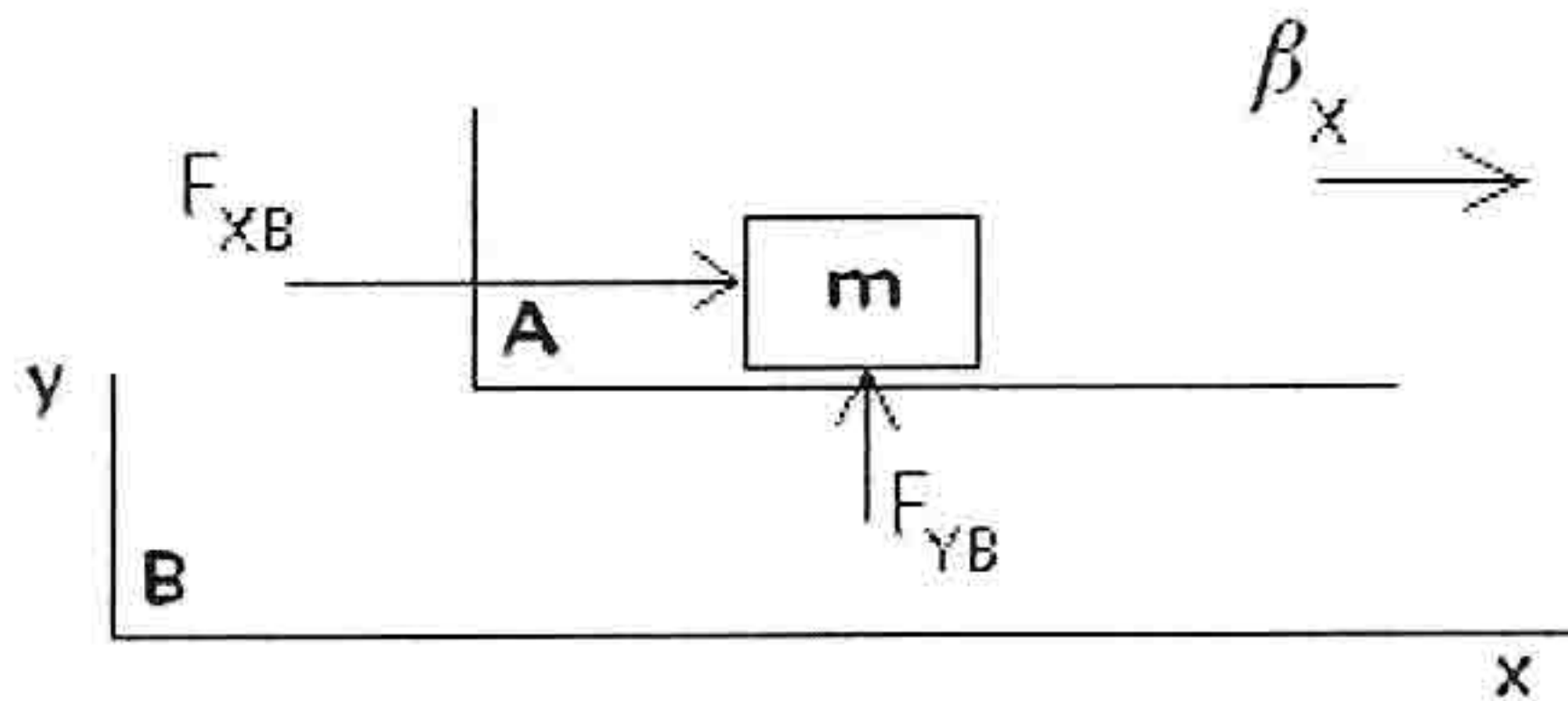


Figure 7. A simple force experiment.

Although the experiment of Figure 7 appears simple, it lacks critical information. The forces F_{xB} and F_{yB} are stationary in frame B but are applied to a moving mass. The analysis of any force experiment will require that the mechanisms of force application be well defined. Force F_{xB} could be applied by a spring mounted to frame B and pressing on moving mass m . But force F_{yB} requires a mechanism that can push perpendicular to the direction of the velocity of mass m . One way to do this is shown in Figure 8.

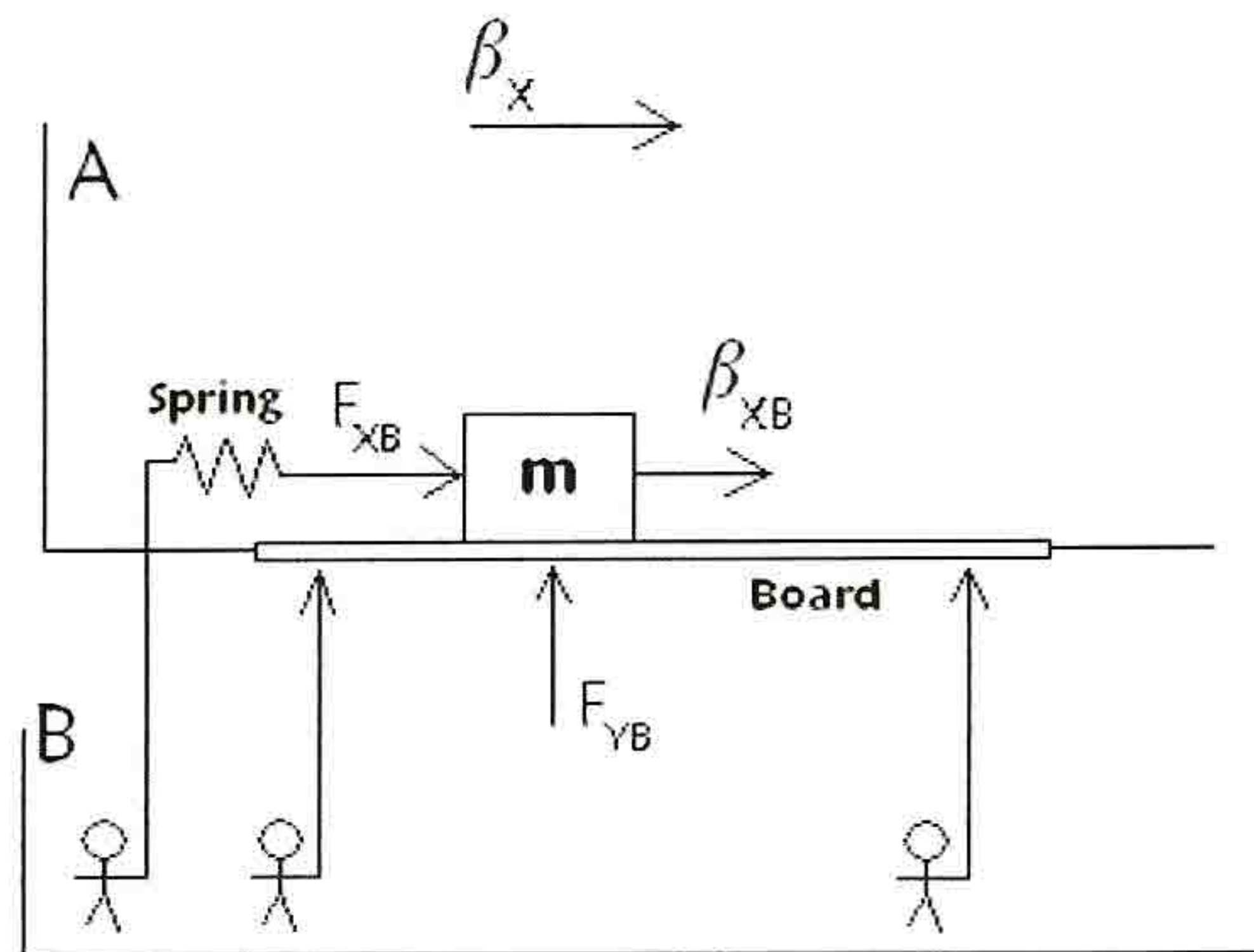


Figure 8. The experiment of Figure 7 with added details of force application.

In Figure 8, two observers are pushing vertically on each end of a board with a frictionless surface that mass m slides across. The observers are careful to keep the board parallel to the x-axis at all times and they are careful to make sure that the total force on mass m is a constant F_{YB} . Mass m is moving with velocity $\beta_{XB} = \beta_x$ and $\beta_{YB} = \beta_y = 0$ relative to frame B (mass m is stationary in frame A). No momentum change of frame B occurs, as a mirror image experiment is run simultaneously in the opposite direction, but not shown.

It is desired to find the relationship between the forces applied by frame B and the forces that mass m “feels”, which would be the forces measured by a reference frame in which mass m is stationary. This happens to be frame A. Applying (20):

$$F_{XB} = \frac{mcA_{XB}}{(1 - \beta_{XB}^2)^{3/2}} \quad (21a)$$

$$F_{YB} = \frac{mcA_{YB}}{\sqrt{1 - \beta_{XB}^2}} \quad (21b)$$

Applying the conditions of this experiment to (16) and (17):

$$A_{XB} = A_{XA}(1 - \beta_{XB}^2)^{3/2} \quad (22a)$$

$$A_{YB} = A_{YA}(1 - \beta_{XB}^2) \quad (22b)$$

The forces that mass m “feels” are the forces observed in reference frame A. These forces are $F_{XA} = mcA_{XA}$ and $F_{YA} = mcA_{YA}$. The final result is:

$$F_{XB} = F_{XA} \quad (23a)$$

$$F_{YB} = F_{YA}\sqrt{1 - \beta_{XB}^2} \quad (23b)$$

The result (23) gives transformations for forces applied between inertial reference frames where mass m is stationary in frame A. Equation (23) is a principle of Special Relativity. It joins the other three principles (length contraction, time dilation and failure of simultaneity at a distance) which explain the fundamental workings of Special Relativity. This principle will be called “Force Enhancement”, due to the way y-directed forces are increased when they pass from the stationary frame B to the moving frame A.

Geometry Distortion and Force Application

The case where there is a frame A y-direction velocity to mass m is worthy of

investigation. In Figure 9, the experiment is the same as that of Figure 8 except that mass m is moving with velocity $\vec{\beta}_B$. Also note that force F_{XB} has now been set to zero.

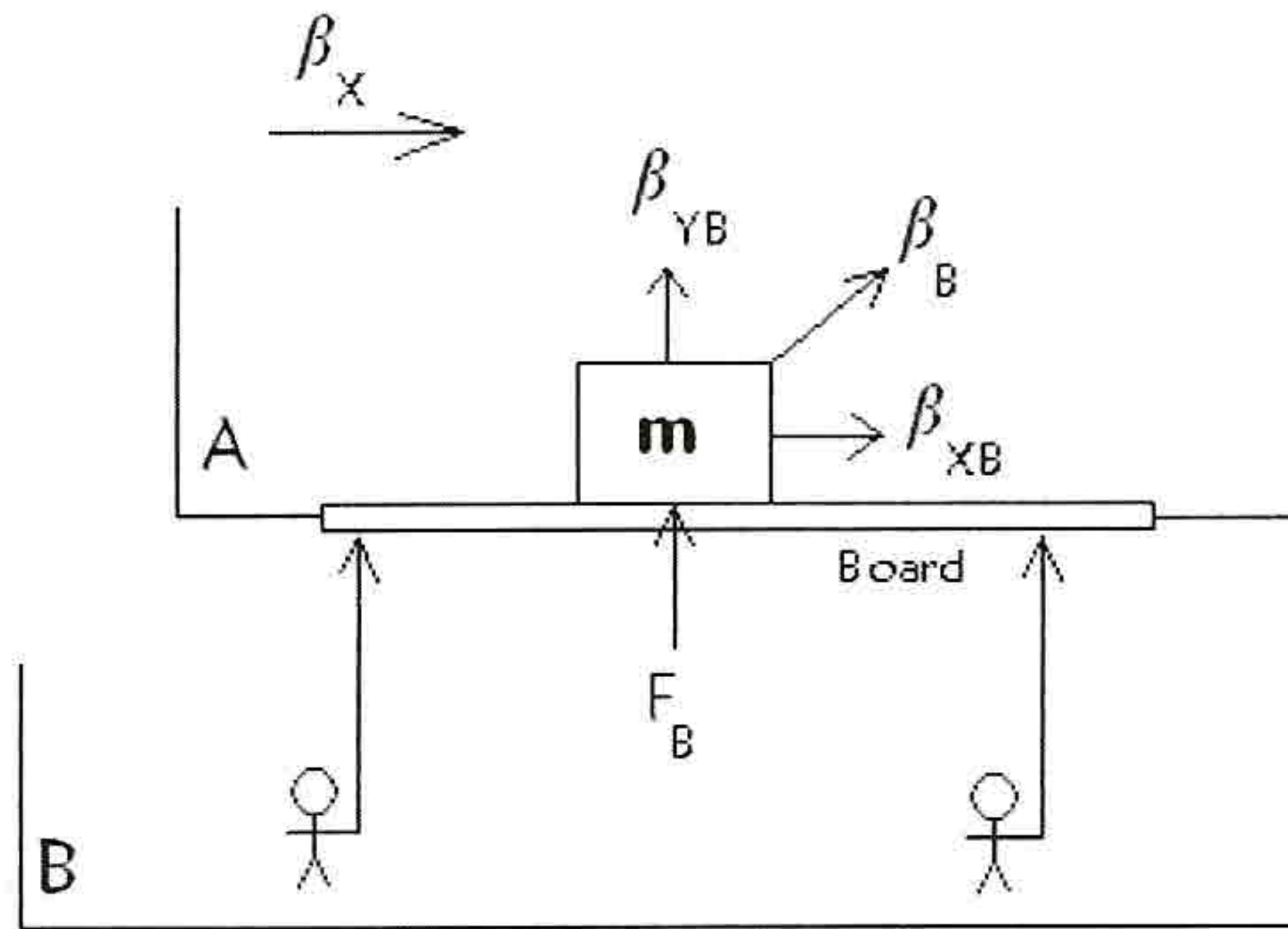


Figure 9. Experiment with mass m moving in frame A.

Frame B sees mass m move in response to any impressed force as described by (20). In this case, (20a) gives:

$$A_{XB} = -\frac{\beta_{XB}\beta_{YB}A_{YB}}{1-\beta_{YB}^2} \quad (24)$$

The frame B observers see an acceleration of mass m in the x-direction, even though the experiment is specifically structured without an x-directed force. They also see mass m move vertically in response to their impressed force F_B . Inserting (24) into (20b) gives the reaction of mass m as seen by frame B.

$$F_B = \frac{mcA_{YB}}{\sqrt{1-\beta_{XB}^2-\beta_{YB}^2}(1-\beta_{YB}^2)} \quad (25)$$

To assist in analyzing what happens in frame A, frame M in which mass m is stationary will be added to the experiment. Frame M moves upward at velocity β_{YA} relative to frame A, as is shown in Figure 10.

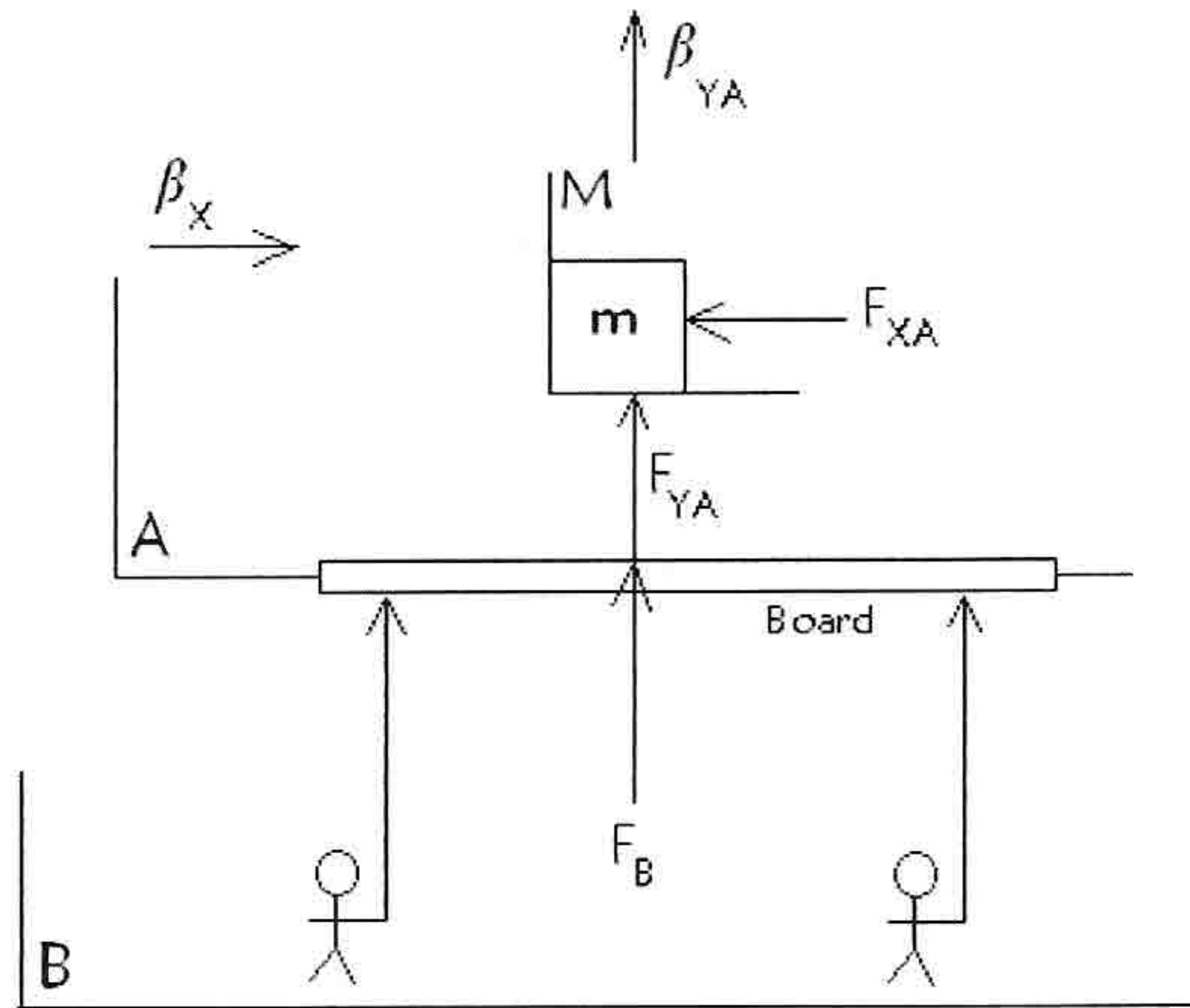


Figure 10. Mass m is stationary in reference frame M .

In Figure 10, the frame A velocity is $\beta_x = \beta_{xB}$, so that $\beta_{xA} = 0$ for mass m . Mass m is stationary in frame M at the instant shown. It is now known from (24) that frame A sees a “mysterious force” F_{XA} applied to mass m . How the board applies this force will be explained later. For now, the relationship of the forces in Figure 10 can be written down by adapting (20) to the frame A point of view.

$$F_{YA} = \frac{mcA_{YA}}{(1 - \beta_{YA}^2)^{3/2}} \quad (26a)$$

$$F_{XA} = \frac{mcA_{XA}}{\sqrt{1 - \beta_{YA}^2}} \quad (26b)$$

The relationship between β_{YA} and β_{YB} can be found from (13).

$$\beta_{YA} = \frac{\beta_{YB}}{\sqrt{1 - \beta_{XB}^2}} \quad (27)$$

The relationship between A_{XA} and A_{XB} can be found from (16).

$$A_{XB} = A_{XA} (1 - \beta_{XB}^2)^{3/2} \quad (28)$$

The relationship between A_{YA} and A_{YB} can be found from (17) and (24),

$$A_{YA} = A_{YB} \frac{1 - \beta_{XB}^2 - \beta_{YB}^2}{(1 - \beta_{XB}^2)^2 (1 - \beta_{YB}^2)} \quad (29)$$

Inserting (27) and (28) into (26b):

$$F_{XA} = \frac{mcA_{XB}}{\sqrt{1 - \beta_{XB}^2 - \beta_{YB}^2} (1 - \beta_{XB}^2)} \quad (30)$$

Inserting (27) and (29) into (26a):

$$F_{YA} = \frac{mcA_{YB}}{\sqrt{1 - \beta_{XB}^2 - \beta_{YB}^2} (1 - \beta_{YB}^2) \sqrt{1 - \beta_{XB}^2}} \quad (31)$$

And:

$$\frac{F_{XA}}{F_{YA}} = \frac{-\beta_{XB} \beta_{YB}}{\sqrt{1 - \beta_{XB}^2}} \quad (32)$$

In this example, there is a horizontal force observed by frame A, even though frame B only applies a vertical force. So, where does “mysterious force“ F_{XA} come from? The answer is that frame B applies this force. The way this occurs is shown by viewing the experiment from the point of view of frame A, as is shown in Figure 11.

Frame A sees frame B going by with velocity $-\beta_{XB}$. The distance between the two observers impressing the vertical force on the board is L and frame A sees this distance as $L\sqrt{1 - \beta_{XB}^2}$. The board has velocity β_{YB} as seen by frame B. The observer that is on the right in Figure 11 is observed by frame A to have a clock that reads “later” than the left observer clock by $L\beta_{XB}/c$. In this time period, the board travels distance $L\beta_{YB}\beta_{XB}$ as seen by either frame. So the right observer’s section of the board has gone up further than the left observers section. This effect is engaged linearly down the length of the board so that the board is tilted as observed by frame A. The force applied by the board is perpendicular to the board surface as seen by either frame. Simple geometry gives:

$$\frac{F_{XA}}{F_{YA}} = \frac{-L\beta_{XB}\beta_{YB}}{L\sqrt{1-\beta_{XB}^2}} = \frac{-\beta_{XB}\beta_{YB}}{\sqrt{1-\beta_{XB}^2}} \quad (33)$$

This is the same result as (32).

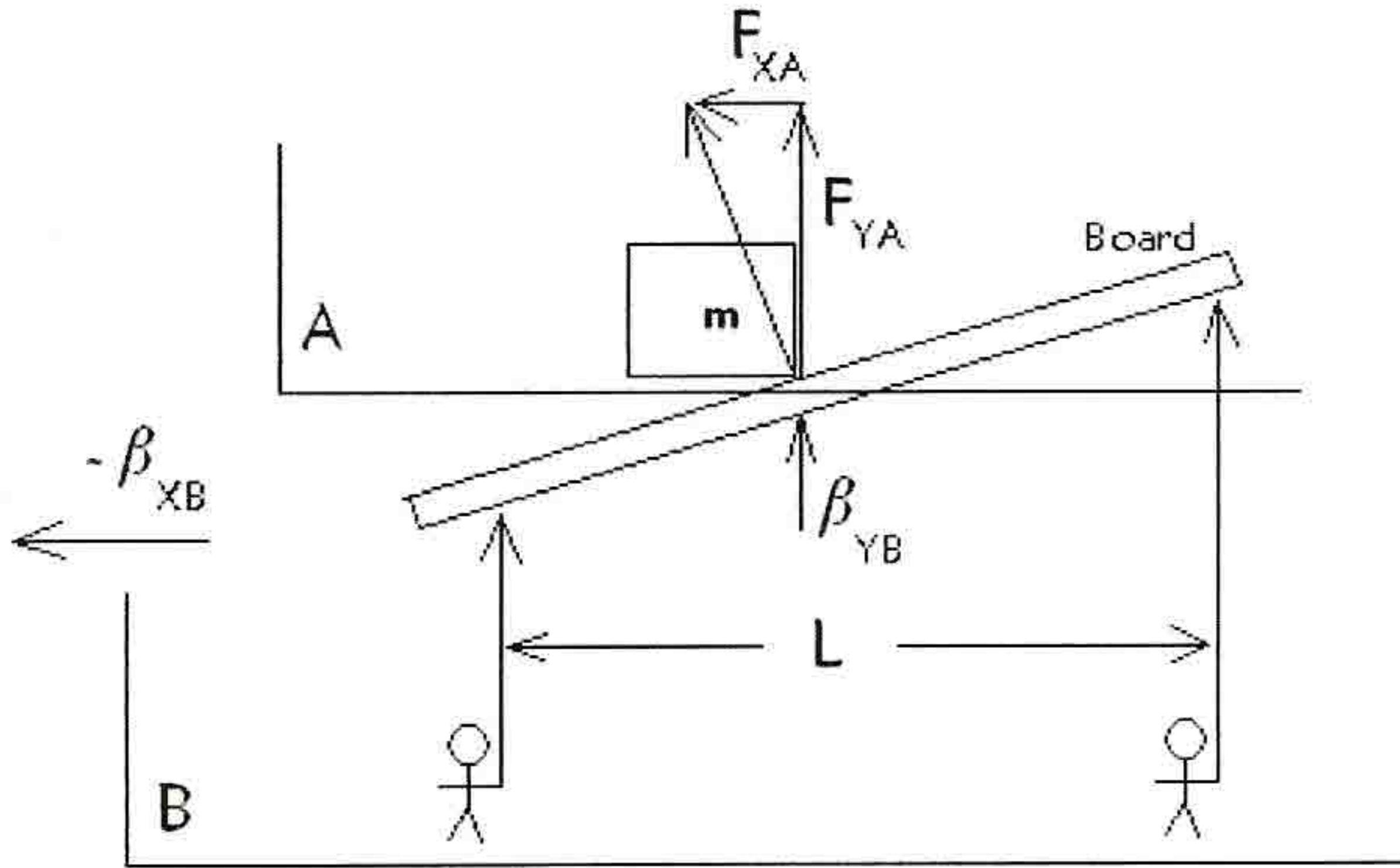


Figure 11. The experiment of Figure 10 as seen by reference frame A.

The space-time geometry of the experiment as viewed by frame A shows how a force perpendicular to the board surface can result in the appearance of a force in the x-direction. Because mass m is not stationary in frame A, this is only an approximate solution. It is reasonably accurate for low values of β_{XB} and β_{YB} , but loses accuracy as those velocities approach light speed. For an exact expression of (32), it is necessary to define frame A as one in which mass m is stationary. In that case:

$$\begin{aligned} \beta_X &= \beta_{XB} & \beta_Y &= \beta_{YB} & \beta_T^2 &= \beta_X^2 + \beta_Y^2 \\ F_{XA} &= mcA_{XA} & F_{YA} &= mcA_{YA} \end{aligned} \quad (34)$$

Equation (24) can be inserted into factor K_B used in (16) and (17) to give:

$$K_B = A_{YB} \frac{\beta_Y - \frac{\beta_X \beta_{XB} \beta_{YB}}{1 - \beta_{YB}^2}}{1 - \beta_X \beta_{XB} - \beta_Y \beta_{YB}} \quad (35)$$

